Problem Sheet 5

1. Let $B = (B^1, B^2, B^3)$ be a Brownian motion in $\mathbb{R}^3$ and $B_0 = y \in \mathbb{R}^3 \backslash \{0\}$.

   (a) Show that $1/||B_t||$ is a local martingale, where
   $$||(x_1, x_2, x_3)|| = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

   (b) Let $A = B(0, r)$ where $r < ||y||$. Show that
   $$\mathbb{P}(\exists t \geqslant 0, B_t \in A) < 1$$
   and comment on this result.

   (c) Let $M_t = ||B_{t+1} - y||^{-1}$. For $t > 0$ show that $\mathbb{E}(M_t^2) = \frac{1}{t+1}$. Deduce that M is bounded in L^2 and uniformly integrable.

   (d) Show that M is both a local martingale and a supermartingale.

   (e) Show that M is *not* a martingale.

2. Let Z be a complex Brownian motion starting at $Z_0 = 0$. Show that $|Z_t|$ does not converge almost surely as $t \to \infty$.

3. Let Z be a complex Brownian motion with $Z_0 = 1$. Let θ be a continuous choice of the argument about 0 of Z. Show that $\limsup_{t\to\infty} \theta(t) = \infty$ and $\liminf_{t\to\infty} \theta(t) = -\infty$.

4. Let Z be a complex Brownian motion started at $i \in \mathbb{H} = \{z \in \mathbb{C}\,;\, \Im z > 0\}$ (the upper half plane) and let $D = \{z \in \mathbb{C}\,;\, |z| < 1\}$ (the unit disc).

   (a) Show that the Moebius transformation $f(z) = \frac{z-i}{z+i}$ maps $\mathbb{H} \to D$, $\partial\mathbb{H} \to \partial D \backslash \{1\}, i \to 0$ and $\infty \to 1$.

   (b) Let $T_{\mathbb{H}} = \inf\{t > 0\,;\, Z_t \notin \mathbb{H}\}$. Show that there exists a time change τ such that $W_t = f(Z_{\tau(t)})$ is a complex Brownian motion and $\tau(S) = T_{\mathbb{H}}$ where $S = \inf\{t \geqslant 0\,;\, W_t \notin D\}$.

   (c) Prove that the exit law is given by
   $$\mathbb{P}(Z_{T_{\mathbb{H}}} \leqslant x) = \frac{1}{2\pi} \arg f(x)$$
   and calculate this expression as
   $$\mathbb{P}(Z_{T_{\mathbb{H}}} \leqslant x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)} \text{t}.$$

5. Let $c > 0$ and let (Z_t) be a complex Brownian motion starting from $Z_0 = 0$. Prove that there is a sequence of times (T_n) such that $T_n \to \infty$ almost surely as $n \to \infty$ and $Z_{T_n} \in (-c, c)$.