Problem Sheet 2

1. Let (B_t) be a real Brownian motion with $B_0 = 0$.

   (a) Show, without reference to the bracket process, that

   $$\lim_{n\to\infty} \sum_{i=1}^{n} (B_{t_{i+1}} - B_{t_i})^2 \to t \qquad \text{in } L^2[0,T]$$

   where $D = \{0 = t_0 < t_1 < \ldots, < t_n = t\}$ and $m(D) = \max|t_{i+1} - t_i| \to 0$ as $n \to \infty$.

   (b) Without using Itô's formula, prove that

   $$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 + t.$$

2. Prove Lemma 1.6.6 from the lectures notes.

3. Let (B_t) be a real Brownian motion with $B_0 = 0$.

   (a) Show that $M_t = e^{t/2}\sin(B_t)$ is a martingale.

   (b) Let $b, x, \sigma \in \mathbb{R}$. Suppose that there exists a real valued continuous square integrable stochastic process X such that

   $$X_t = x + \int_0^t bX_s ds + \int_0^t \sigma X_s dB_s$$

   for all $t > 0$. Calculate $\mathbb{E}\left[X_t^2\right]$.

   (c) Find a non-constant function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t + B_t)$ is a martingale. Let $a, b > 0$ and let $\tau = \inf\{s \geqslant 0\,;\, B_t = -a \text{ or } b\}$. Calculate $\mathbb{E}\left[\tau\right]$.

4. Let (B_t) be a real Brownian motion with $B_0 = 0$.

   (a) Show that there exists $\epsilon, \delta > 0$ such that $\mathbb{P}\left[\forall t \in [0,\epsilon], |B_t| < \delta\right] > 0$.

   (b) Does this result extend to Brownian motion in dimensions $d > 1$?