Problem Sheet 3

1. Deduce the complex version of Itô's formula from the real version.

2. Let $Z = X + iY$ be a complex Brownian motion and let $T = \inf\{s \geqslant 0\,;\, |Y_s| = 1\}$. Show that $(Z_{t \wedge T})$ is a square integrable martingale.

3. Prove that a non-constant entire function cannot omit the interval $[0, 1]$ from its range.

4. Let (Z_t) be a complex Brownian motion with $Z_0 = 0$. Let T be a stopping time such that $Z_T = 0$. Define

$$W_t = \begin{cases} Z_t & t \leqslant T \\ \bar{Z}_t & t > T. \end{cases}$$

   Prove that W is a complex Brownian motion.

5. (a) Let $r > 0$ and let $D = \{z \in \mathbb{C}\,;\, |z| < r\}$. Let (Z_t) be a complex Brownian motion with $Z_0 = 0$ and let $T = \inf\{t \geqslant 0\,;\, Z_t \notin D\}$. What is the distribution of Z_T?

   (b) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Show that for all $z \in \mathbb{C}$ and $r > 0$,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta.$$