Problem Sheet 1

1. Let (B_t) be a Brownian motion with $B_0 = 0$ and let $(\mathcal{F}_t)$ be its generated filtration. Let $M_t = B_t^2 - t$. Show that (B_t) and (M_t) are both $(\mathcal{F}_t)$-martingales.

2. Let (B_t) be a Brownian motion with $B_0 = 1$ and define $\tau = \inf\{t \geqslant 0\,;\, B_t = 0\}$. Let $(\mathcal{F}_t)$ be the filtration generated by (B_t).

   (a) Show that τ is an $(\mathcal{F}_t)$-stopping time.

   (b) Show that $\mathbb{E}\left[B_{t\wedge\tau}\right] = 1$ for all $t \geqslant 0$ and that $\mathbb{P}\left[B_\tau = 0\right] = 1$.

   (c) Deduce that $\{B_{t\wedge\tau}\,;\, t \geqslant 0\}$ is not uniformly integrable.

3. Let (N_t) be the pure birth process, started at $N_0 = 1$, in which each currently alive individual gives birth at rate 1. You may assume that (N_t) is a Markov process with state space $\mathbb{N}$ and note that transitions from $n \to n+1$ occur at rate n. Let $(\mathcal{F}_t)$ be the filtration generated by (N_t).

   (a) Show that $W_t = \frac{N_t}{\mathbb{E}[N_t]}$ is an $(\mathcal{F}_t)$-martingale.

   (b) Show that $n_t = \mathbb{E}\left[N_t\right]$ satisfies $\frac{dn_t}{dt} = n_t$ and deduce that $e^{-t}N_t$ converges almost surely as $t \to \infty$ to some random variable L.

4. Let (B_t) be a Brownian motion with $B_0 = 0$ and that is adapted to the filtration $(\mathcal{F}_t)$. Let C be a $(0,\infty)$ valued random variable that is independent of B and $\mathcal{F}_t$ measurable for all t. Let $M_t = B_{tC}$.

   (a) Show that (M_t) is a local $(\mathcal{F}_t)$-martingale.

   (b) Show that (M_t) is an $(\mathcal{F}_t)$-martingale if and only if $\mathbb{E}\left[C^{1/2}\right] < \infty$.

5. Let (B_t) be a Brownian motion with $B_0 = 0$ and let $(\mathcal{F}_t)$ be its generated filtration. Let $S = \inf\{t > 0\,;\, B_t = -1\}$. For $t \in [0,1)$ let $\tau(t) = \frac{t}{1-t}$. Note that $\tau(0) = 0$, τ is strictly increasing on $[0,1)$ and that $\tau(t) \uparrow \infty$ as $t \uparrow 1$. Define

$$M_t = \begin{cases} B_{\tau(t)\wedge S} & t < 1 \\ -1 & t \geqslant 1 \end{cases}.$$

   and let $(\mathcal{G}_t)$ be the filtration defined by $\mathcal{G}_t = \mathcal{F}_{\tau(t)}$ for $t < 1$ and $\mathcal{G}_t = \cup_{s=0}^{\infty}\mathcal{F}_s$ for $t \geqslant 1$.

   (a) Show that (M_t) is a local $(\mathcal{G}_t)$-martingale.

   (b) Show that (M_t) is not an $(\mathcal{G}_t)$-martingale.