Problem Sheet 4

Let W_t be a complex Brownian motion and let T_D be the exit time of W from the unit disc D. Let f be a continuous function on ∂D and define u on $\overline{D}$ by $u = f$ on ∂D and

$$u(z) = \mathbb{E}_z[f(W_{T_D})]$$

in D. You may assume that u is continuous in D (or use the Poisson kernel to prove it).

1. (a) Fix $z \in D \backslash \{0\}$, and let γ be the (unique) circle through z which cuts ∂D twice, both times at right angles. Let J and $\hat{J}$ be the segments of ∂D inside and outside of γ. Prove that

$$\mathbb{P}_z(W_{T_D} \in J) = \mathbb{P}_z(W_{T_D} \in \hat{J}) = \frac{1}{2}.$$

   (b) Show that u is continuous on ∂D.

2. (a) Let $z \in D$ and $\epsilon > 0$ be small so as $\{w \in \mathbb{C}\,;\, |z - w| = \epsilon\} \subseteq D$. Use the strong Markov property (informally) to show that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + \epsilon e^{i\theta})\, d\theta$$

   That is, the average of u around the circle radius ϵ is the value of u at the center.

   (b) Let $\epsilon > 0$. Suppose $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a symmetric smooth function with support in $B_{\mathbb{R}^2}(0, \epsilon)$ such that $\int \phi = 1$. For $\epsilon < \operatorname{dist}(a + ib, \partial D)$, define the convolution

$$(\phi * u)(a, b) = \int_{(s,t)\in\mathbb{R}^2} \phi(s, t) u\big((a, b) - (s, t)\big)\, d(s, t)$$

   identifying $\mathbb{C}$ with $\mathbb{R}^2$ so as $u(z) = u(x + iy) = u(x, y) = (\Re u(z), \Im u(z))$ (extend u by $u = 0$ outside D). You may assume $\phi * u$ is a smooth function. Explain why one might expect $\phi * u(z)$ to approximate $u(z)$ for small ϵ.

   (c) Suppose ϕ is circularly symmetric. Prove that, in fact, for sufficiently small ϵ, $\phi * u(z) = u(z)$. Deduce that u is smooth.

   (d) Use Ito's formula to show that u is harmonic in D.

3. How much of the above remains true if we replace our condition that f be continuous by the weaker condition $\int_0^{2\pi} |f(e^{i\theta})|\, d\theta < \infty$?