Problem Sheet 6

Recall that if $Z = X + iY$ is a complex Brownian motion starting at i, $\mathbb{H}$ is the upper half plane and $T = \inf\{t > 0\,;\, Z_t \notin \mathbb{H}\}$, then $Z_T \in \mathbb{R}$ a.s. and

$$\mathbb{P}(Z_T \leqslant x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\, dt.$$

That is, Z_T has the Cauchy$(0,1)$ distribution.

1. Let Z be a complex Brownian motion started at $z_0 = x_0 + iy_0 \in \mathbb{H}$. Compute the exit law of Z from $\mathbb{H}$.

2. Let $\alpha \in (0, \pi)$ and define $C_\alpha = \{z \in \mathbb{C}\,;\, \arg(z) \in (0, \alpha)\}$. Let Z be a complex Brownian motion started at $Z_0 = z_0 \in C_\alpha$. Let $T = \inf\{t \geqslant 0\,;\, Z_t \notin C_\alpha\}$. Compute the law of Z_T.

3. Let $D = \{z \in \mathbb{C}\,;\, |z| < 1\}$ and Z be a complex Brownian motion started at $z_0 \in (0,1)$. Using the transformation
$$f(z) = \frac{z - z_0}{z z_0 - 1}$$
(which you may assume maps $D \leftrightarrow D$), show that, if $T = \inf\{t > 0\,;\, Z_t \in \partial D\}$ and $\theta_0 \in [0, 2\pi)$,
$$\mathbb{P}(\arg(Z_T) \leqslant \theta_0) = \frac{1}{2\pi} \int_0^{\theta_0} \frac{1 - |z_0|^2}{|z_0 - e^{i\theta}|}\, d\theta$$

4. Let Z be a complex Brownian motion starting at $Z_0 = 1/e$. Let
$$\Theta_t = \frac{1}{2\pi} \Im\left(\int_0^t \frac{1}{Z_s}\, dZ_s\right)$$
denote the winding of Z about 0, and T denote the exit time of Z from the unit disc D. Show that Θ_T has a Cauchy$(0,1)$ distribution.